1. Find the derivative

   (a) $f(x) = e^{2\ln(3x+1)}$ $= e^{\ln(3x+1)^2} = (3x+1)^2$

   $f'(x) = 2(3x+1)^1(3)$ or $18x+6$

   (b) $f(x) = 5x^{-2} - [\ln\cos x - \ln(\sin x + x)]$

   $\frac{-10}{x^3} - \left[\frac{-\sin x}{\cos x} - \frac{\cos x + 1}{\sin x + x}\right]$

   (c) $f(x) = \dfrac{e^x + 9}{e^{x^2} - x^4}$

   $f'(x) = \dfrac{(e^{x^2} - x^4)(e^x) - (2xe^{x^2} - 4x^3)(e^x+9)}{(e^{x^2} - x^4)^2}$

   (d) $f(x) = \ln(2x^2 + 1)$

   $f'(x) = \dfrac{4x}{2x^2+1}$

   (e) $y = x^{\sqrt{2}}$

   $y' = \sqrt{2}\, x^{\sqrt{2}-1}$

   (f) $y = x^x$

   $\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$

   $\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + 1 \cdot \ln x$

   $\frac{dy}{dx} = y(1 + \ln x)$

   $y' = x^x(1 + \ln x)$

   (g) $f(x) = \dfrac{\sec x}{x}$

   $f'(x) = \dfrac{x(\sec x \tan x) - 1 \cdot \sec x}{x^2}$

(h) $\ln y + [xy^2] - 4x^3 + 10 = 3x$ P.R.

$\frac{1}{y}\frac{dy}{dx} + \left[x(2y)\frac{dy}{dx} + 1\cdot y^2\right] - 12x^2 + 0 = 3$

$\frac{dy}{dx}\left[\frac{1}{y} + 2xy\right] = 3 - y^2 + 12x^2$

$\frac{dy}{dx} = \frac{12x^2 - y^2 + 3}{\frac{1}{y} + 2xy}$ or $\frac{12x^2y - y^3 + 3y}{1 + 2xy^2}$

(i) $f(x) = (x^2 + 6)\ln(3x)$

$f'(x) = (x^2+6)\cdot\frac{1}{3x}\cdot 3 + 2x\ln 3x$

or $\frac{3(x^2+6)}{3x} + 2x\ln 3x$

or $x + \frac{6}{x} + 2x\ln 3x$

(j) $f(x) = \cot x$

or $\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$

$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$

(k) $f(x) = x^{\tan x}$

$y = x^{\tan x}$

$\ln y = \tan x \ln x$

$\frac{1}{y}y' = \frac{\tan x}{x} + \sec^2 x \ln x$

$y' = x^{\tan x}\left(\frac{\tan x}{x} + \sec^2 x \ln x\right)$

or $x^{\tan x - 1}\tan x + x^{\tan x}\sec^2 x \ln x$

or $x^{\tan x - 1}(\tan x + x\sec^2 \ln x)$ etc

(l) $y = \cos x(\tan x - \sec x)$ $= \sin x - 1$

$y' = \cos x$

(m) $f(x) = 3^{4x}$

$f'(x) = 3^{4x} \cdot \ln 3 \cdot 4$

or $4 \ln 3 \cdot 3^{4x}$

or $\ln 81 \cdot 3^{4x}$

or if your section 5.5 is rusty:

$y = 3^{4x}$

$\ln y = 4x \ln 3$

$\frac{1}{y} y' = 4 \ln 3$

$y' = 3^{4x} \cdot 4 \ln 3$

(n) $f(t) = \dfrac{3^{2t}}{t}$

$$f'(t) = \frac{t(3^{2t} \cdot \ln 3 \cdot 2) - 1 \cdot (3^{2t})}{t^2}$$

or $\dfrac{3^{2t} \ln 9}{t} - \dfrac{3^{2t}}{t^2}$

or $9^t \left(\dfrac{\ln 9}{t} - \dfrac{1}{t^2} \right)$ etc...

(o) $y = \log_5 \dfrac{x^2 - 1}{x}$ $= \log_5 x^2 - 1 - \log_5 x$

$$y' = \frac{1}{x^2-1} \cdot \frac{1}{\ln 5} \cdot 2x - \frac{1}{x} \cdot \frac{1}{\ln 5}$$

or $\dfrac{1}{\ln 5} \left(\dfrac{2x}{x^2-1} - \dfrac{1}{x} \right)$

or $\dfrac{1}{\ln 5} \left(\dfrac{2x^2 - (x^2-1)}{(x^2-1)(x)} \right)$ or $\dfrac{1}{\ln 5} \left(\dfrac{x^2+1}{x^3-x} \right)$ etc...

(p) $g(t) = \log_2(t^2 + 7)^3$ $= 3 \log_2 (t^2+7)$

$$g'(t) = 3 \cdot \frac{1}{t^2+7} \cdot (2t) \cdot \frac{1}{\ln 2}$$

or $\dfrac{6t}{\ln 2 (t^2+7)}$

2. Evaluate the integral.

(a) $\int e^{\sec 2x} \sec 2x \tan 2x \; dx$ $= \frac{1}{2}\int e^u \, du$

$\boxed{\begin{aligned} u &= \sec 2x \\ du &= 2\sec 2x \tan 2x \, dx \\ \tfrac{1}{2} du &= \sec 2x \tan 2x \, dx \end{aligned}}$

$= \frac{e^{\sec 2x}}{2} + C$

(b) $\int \sec y(\tan y - \sec y) \; dy$ $= \int \frac{\sin y}{\cos^2 y} dy - \int \sec^2 y \, dy$

$\boxed{\begin{aligned} u &= \cos y \\ du &= -\sin y \, dy \\ -du &= \sin y \, dy \end{aligned}}$

$-\int u^{-2} \, du - \tan y + C$

$-\frac{(\cos y)^{-1}}{-1} - \tan y + C$

$\sec y - \tan y + C$

(c) $\int e^{3x} \; dx$

$\boxed{\begin{aligned} u &= 3x \\ du &= 3 \, dx \\ \tfrac{1}{3} du &= dx \end{aligned}}$

$\frac{1}{3}\int e^u \, du = \frac{1}{3} e^{3x} + C$

(d) $\int \tan^2 x + 1 \; dx$ $= \int \sec^2 \, dx$

$\boxed{\text{Recall } \frac{\sin^2}{\cos^2} + \frac{\cos^2}{\cos^2} = \frac{1}{\cos^2}}$

$\tan x + C$

(e) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}}$ $= \int u^2\,du = \frac{1}{3}u^3 + C = \frac{(\ln x)^3}{3} + C$

(f) $\displaystyle\int \frac{x}{\sqrt{2x-1}}\,dx$ $= \frac{1}{2}\int u^{-1/2}\,\frac{(u+1)}{2}\,du$

$\boxed{\begin{array}{l} u = 2x-1 \\ du = 2\,dx \\ \frac{1}{2}du = dx \\ x = \frac{u+1}{2} \end{array}}$

$= \frac{1}{4}\int u^{1/2} + u^{-1/2}\,du$

$= \frac{1}{4}\left[\frac{2}{3}u^{3/2} + 2u^{1/2}\right] + C$

$= \frac{1}{6}(2x-1)^{3/2} + \frac{1}{2}\sqrt{2x-1} + C$

or $\frac{\sqrt{2x-1}}{6}(2x-1+3) + C$

or $\frac{1}{3}\sqrt{2x-1}\,(x+1) + C$

(g) $\displaystyle\int \frac{1}{3x+2}\,dx$

$\boxed{\begin{array}{l} u = 3x+2 \\ du = 3\,dx \\ \frac{1}{3}du = dx \end{array}}$ $= \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|3x+2| + C$

(h) $\displaystyle\int \cot x\,dx$ $= \ln|\sin x| + C$ (see p 329)

or $\displaystyle\int \frac{\cos x}{\sin x}\,dx$

$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x\,dx \end{array}}$

$\int \frac{1}{u}\,du$

$= \ln|u| + C$

$= \ln|\sin x| + C$

(i) $\int \frac{12}{1+9x^2}\,dx$ $= 4\int \frac{1}{1+u^2}\,du = \frac{4}{1}\arctan\frac{u}{1} + C$

$\boxed{\begin{array}{l} u = 3x \\ du = 3\,dx \\ a = 1 \end{array}}$

$= 4\arctan 3x + C$

(j) $\int \frac{1}{\sqrt{-x^2-4x}}\,dx$ $-(x^2+4x+4)+4 = 4-(x+2)^2$

Hint: Complete the square

$u = x+2$

$du = dx$

$\int \frac{1}{\sqrt{4-u^2}}\,du = \arcsin\frac{u}{2} + C = \arcsin\frac{x+2}{2} + C$

$a = 2$

(k) $\int \frac{e^{2y}}{1-e^{2y}}\,dy$

$\boxed{\begin{array}{l} u = 1-e^{2y} \\ du = -2e^{2y}\,dy \\ -\frac{1}{2}du = e^{2y}\,du \end{array}}$

$-\frac{1}{2}\int \frac{1}{u}\,du$

$-\frac{1}{2}\ln|u| + C$

$-\frac{1}{2}\ln|1-e^{2y}| + C$

(l) $\int \frac{e^{3x}-2e^x+5}{e^{2x}}\,dx$

$= \int e^{3x-2x} - 2e^{x-2x} + 5e^{-2x}\,dx$

$= \int e^x - 2e^{-x} + 5e^{-2x}\,dx$

$= e^x + 2e^{-x} + -\frac{5}{2}e^{-2x} + C$

$5\int e^{-2x}\,dx$

$\boxed{\begin{array}{l} u = -2x \\ du = -2dx \\ -\frac{1}{2}du = dx \end{array}}$

$= -\frac{5}{2}e^u + C = -\frac{5}{2}e^{-2x} + C$

(m) $\int 2^x\,dx$

$2^x \cdot \frac{1}{\ln 2} + C = \frac{2^x}{\ln 2} + C$

(n) $\int_1^3 4^{x+1} + 2^x \, dx$

$$\frac{4^{x+1}}{\ln 4} + \frac{2^x}{\ln 2}\Big|_1^3 = \frac{4 \cdot 4^x}{2 \cdot \ln 2} + \frac{2^x}{\ln 2}\Big|_1^3 = \frac{2 \cdot 4^x + 2^x}{\ln 2}\Big|_1^3 = \frac{1}{\ln 2}\left[(128+8) - (8+2)\right]$$

$$= \frac{126}{\ln 2}$$

(o) $\int_1^3 \frac{e^{3/x}}{x^2} \, dx$ $= -\frac{1}{3}\int_3^1 e^u \, du = \frac{1}{3}\int_1^3 e^u \, du = \frac{1}{3} e^u \Big|_1^3$

$= \frac{e^3 - e^1}{3}$

$u = \frac{3}{x}$

$du = \frac{-3}{x^2} dx$

$-\frac{1}{3} du = \frac{1}{x^2} dx$

$u(1) = 3$

$u(3) = 1$

(p) $\int_0^{\sqrt{2}} \frac{1}{\sqrt{4 - x^2}} \, dx$ $= \arcsin \frac{x}{2}\Big|_0^{\sqrt{2}} = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

$a = 2$

(q) $\int_{-2}^3 \frac{1}{x^2 + 4x + 8} \, dx$ $= \int_{-2}^3 \frac{1}{(x^2+4x+4)+4} dx = \int_{-2}^3 \frac{1}{(x+2)^2 + 2^2} dx$

Hint: Complete the square

$\boxed{\begin{array}{l} a = 2 \\ u = x+2 \\ u(3) = 5 \\ u(-2) = 0 \end{array}}$

$\frac{1}{2}\arctan \frac{u}{2}\Big|_0^5$

$$\frac{1}{2}\arctan \frac{x+2}{2}\Big|_{-2}^3 = \frac{1}{2}\left[\operatorname{atan}\frac{5}{2} - \operatorname{atan} 0\right] = \frac{1}{2}\arctan\frac{5}{2}$$

(r) $\int_0^{\pi/2} \frac{\cos x}{2^{\sin x}} \, dx$ $= \int_0^1 2^{-u} \, du =$

$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x \, dx \\ u(0) = 0 \\ u(\frac{\pi}{2}) = 1 \end{array}}$

$$\frac{-1}{\ln 2} 2^{-u}\Big|_0^1 = \frac{-1}{\ln 2}\left(\frac{1}{2} - 1\right) = \frac{1/2}{\ln 2} = \frac{1}{2\ln 2} = \frac{1}{\ln 4}$$

$\int_0^1 \left(\frac{1}{2}\right)^u du$

$$\frac{1}{\ln \frac{1}{2}} \cdot \left(\frac{1}{2}\right)^u\Big|_0^1 = \frac{1}{\ln \frac{1}{2}}\left(\frac{1}{2} - 1\right) = \frac{-1/2}{\ln \frac{1}{2}} = \frac{1/2}{\ln 2} = \frac{1}{2\ln 2} = \frac{1}{\ln 2^2} = \frac{1}{\ln 4}$$

$(\approx .721)$

3. Evaluate the limits, using L'Hôpital's Rule if necessary. If you do, remember to identify if it is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and state that you are using L'Hôpital's Rule.

(a) $\lim_{x\to-3} \frac{3\sin(2x+6)}{3+x}$ $\boxed{\frac{0}{0} \text{ form, by L'Hop}} = \lim_{x\to-3} \frac{3(2)\cos(2x+6)}{1} = 6$

(b) $\lim_{x\to3} \frac{3\ln(4-x)}{x-3}$ $\boxed{\frac{0}{0} \text{ form, by L.H.}} = \lim_{x\to3} \frac{\frac{3}{4-x}(-1)}{1} = -3$

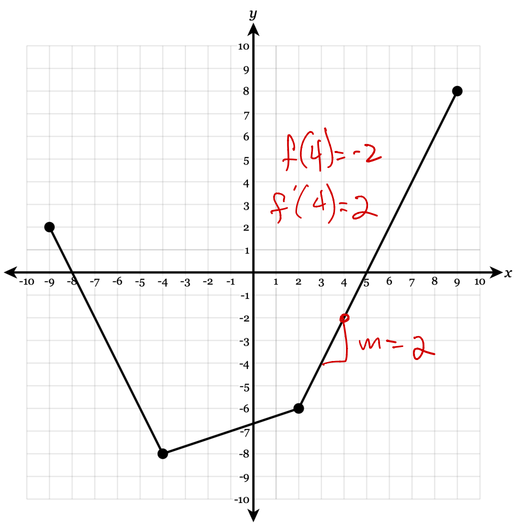
(c) $\lim_{x\to\infty} \frac{\arctan x}{3} = \frac{\frac{\pi}{2}}{3} = \pi/6$ (Do not use L.H.)

(d) $\lim_{x\to2} \frac{x^2-4}{x+2} = \frac{0}{4} = 0$ (Do not use L.H)

(e) $\lim_{x\to\infty} \frac{\ln x^2}{(\ln x)^2}$ $\boxed{\frac{\infty}{\infty} \text{ form, by L.H}} = \lim_{x\to\infty} \frac{\frac{1}{x^2}\cdot 2x}{2\ln x \cdot \frac{1}{x}} = \lim_{x\to\infty} \frac{\cancel{\frac{2}{x}}}{\cancel{\frac{2}{x}}\ln x} = 0$

(f) $\lim_{x\to\infty} \frac{\ln 6x}{\ln 2x}$ $\boxed{\frac{\infty}{\infty} \text{ form, by L.H.}} = \lim_{x\to\infty} \frac{6\cdot\frac{1}{6x}}{2\cdot\frac{1}{2x}} = 1$

4. The graph of the function f is shown below. Determine the value of $\lim_{x\to2} \frac{f(2x)+2}{5x-10}$



$f(4)+2 = -2+2 = 0$
$5(2)-10 = 0$ $\Big\}$ $\frac{0}{0}$ form so by L'Hôpital:

$= \lim_{x\to2} \frac{f'(2x)\cdot 2}{5}$ ← remember Chain Rule!

$f'(4) = 2$

So $\lim_{x\to2} \frac{f'(2x)}{5} = \frac{4}{5}$

(See Practice L'Hôpital at Delta Math for more like this)

5. Find an equation of the tangent line to $y = 5^{x-2}$ at the point $(2, 1)$

$y' = (\ln 5)(5^{x-2})(1)$

$y'(2) = (\ln 5)(5^0) = \ln 5$

$y - 1 = \ln 5(x-2)$

6. If $f(x) = \int_{\arctan x}^{2} 7^t \, dt$, then find $f'(x)$. (*Hint: FTC2 and the chain rule*)

$f(x) = -\int_2^{\arctan x} 7^t \, dt$ by 2^{nd} FTC

$f'(x) = -7^{\arctan x}\left(\frac{1}{x^2+1}\right)$ or $\frac{-7^{\arctan x}}{x^2+1}$

7. (Calculator Active) The weight (in grams) of a bacterial culture at time t (hours) is modeled by the function

$$W(t) = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

for time $t \geq 0$

(a) Find the weight after 1 hour.

$W(1) = 1.070590441$ grams

(b) Find the rate at which the weight is increasing after 2 hours. (increasing)

$\frac{d}{dt} W(t)\Big|_{t=2} = .045394724 2$ grams per hour

(use $\boxed{\text{MATH}}\boxed{\text{8. nDeriv}}$ on TI)

8. (Calculator Active) At what point (x, y) on the graph of $y = 2^x - 3$ does the tangent line have slope 21?

$y' = \ln 2 \cdot 2^x = 21$

$2^x = \frac{21}{\ln 2}$

$x = \log_2\left(\frac{21}{\ln 2}\right)$

$x = \frac{\ln\left(\frac{21}{\ln 2}\right)}{\ln 2} \approx 4.921083796$

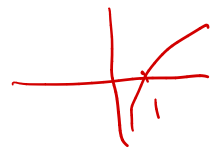
point is $(4.921, 27.296)$ or $(4.921, 27.297)$

9. (No Calculator) A particle moves along the x axis so that at any time $t > 0$ its velocity is given by $v(t) = t\ln t - t$ At time $t = 1$, the position of the particle is $x(1) = 6$.

product rule

(a) Write an expression for the acceleration of the particle.

$$a(t) = v'(t) = (t)\left(\frac{1}{t}\right) + (1)(\ln t) - 1$$
$$= 1 + \ln t - 1$$
$$= \ln t$$



(b) For what values of t is the particle moving right?

when $v(t) > 0$

$t \ln t - t > 0$

$t(\ln t - 1) > 0$

given $t > 0$, so when $\ln t - 1 > 0$

$\ln t > 1$

$t > e^1$

(c) What is the minimum velocity of the particle. Justify your conclusion.

v' ⊖ ⊕

1

so $v'(t)$ changes from neg. to pos at $t = 1$

so min velocity is $1(\ln 1 - 1) = 0 - 1 = -1$

(d) If $\int t\ln t - t\;dt = \frac{1}{4}t^2(2\ln t - 3) + C$, write an expression of the position $x(t)$ of the particle.

$x(1) = 6$ so

$$\frac{1}{4}1^2(2\ln 1 - 3) + C = 6$$
$$\frac{1}{4}(-3) + C = 6$$
$$C = 6 + \frac{3}{4} = \frac{24+3}{4} = \frac{27}{4}$$

$$x(t) = \frac{1}{4}t^2(2\ln t - 3) + \frac{27}{4}$$

10. (No Calculator) Let $f(x) = e^x \cos x$.

(a) (1 point) Find the average rate of change of f on the interval $0 \le x \le \pi$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{e^{\pi}\cos\pi - e^{0}\cos 0}{\pi}$$

$$= \frac{e^{\pi}(-1) - 1}{\pi} \quad \left(\text{or } -\frac{e^{\pi}}{\pi} - \frac{1}{\pi}\right)$$

(b) (2 points) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

$$f'(x) = e^{x}(-\sin x) + e^{x}\cos x = e^{x}(\cos x - \sin x)$$

$$f'\left(\tfrac{3\pi}{2}\right) = e^{3\pi/2}\left(\cos\tfrac{3\pi}{2} - \sin\tfrac{3\pi}{2}\right)$$

$$= e^{3\pi/2}\big(0 - (-1)\big) = e^{3\pi/2}$$

Candidate test

(c) (3 points) Find the absolute minimum value of f on the interval $0 \le x \le 2\pi$. Justify your answer.

$$0 = f'(x) = e^{x}(\cos x - \sin x)$$

$$\cos x = \sin x$$

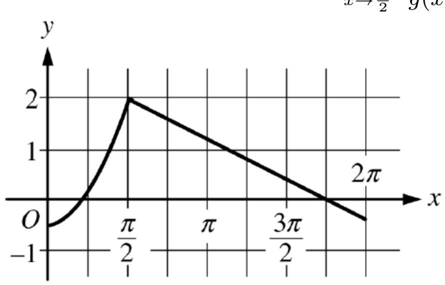
$$x = \frac{\pi}{4}, \frac{5\pi}{4} \quad (\text{when}$$

x	0	$\pi/4$	$5\pi/4$	2π
$f(x)$	1	$e^{\pi/4}\cdot\frac{\sqrt{2}}{2}$	$e^{5\pi/4}\cdot\frac{-\sqrt{2}}{2}$	$e^{2\pi}$

Neg, Ab Min by Candidate Test

(d) (3 points) Let g be a differentiable function such that $g(\frac{\pi}{2}) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim_{x\to\frac{\pi}{2}} \frac{f(x)}{g(x)}$

and therefore continuous



Graph of g'

$$f\left(\tfrac{\pi}{2}\right) = e^{\pi/2}\cdot\cos\tfrac{\pi}{2} = 0$$

$$g\left(\tfrac{\pi}{2}\right) = 0 \ (\text{given}) \quad \text{so } \tfrac{0}{0} \text{ form}$$

$$\text{by L'H: } \lim_{x\to\frac{\pi}{2}} \frac{f'(x)}{g'(x)} = \frac{e^{\pi/2}\left(\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right)}{2}$$

$$= \frac{-e^{\pi/2}}{2}$$